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Quantum Tunneling of Spinor Bose–Einstein Condensates in an Optical Lattice

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Abstract We have studied tunneling of spinor Bose–Einstein condensate in an optical lattice. It is found that, when the system being prepared in a squeezed coherent state, there exist the quantum tunneling between lattices l and $l+1$, l and $l-1$, respectively. In particular, when the optical lattice is infinitely long and the spin excitations are in the long-wavelength limit, quantum tunneling disappear between lattices l and $l+1$, and that l and $l-1$, in this case the magnetic soliton appears.

Keywords Spinor Bose–Einstein condensate · Quantum tunneling

1 Introduction

Recent advance of experimental techniques on Bose–Einstein condensate (BEC) prompts us to closely and seriously look into theoretical possibilities which were mere imagination for theoreticians in this field. This is particularly true for spinor BEC where all hyperfine states of an atom Bose-condensed simultaneously, keeping these "spin" states degenerate and active. Recently, Barrett et al. [1] have succeeded in cooling Rb_{87} with the hyperfine state $F=1$ by all optical methods without resorting to a usual magnetic trap in which the internal degrees of freedom is frozen. Since the spin interaction of the Rb_{87} atomic system is ferromagnetic, based on the refined calculation of the atomic interaction parameters by Klausen et al. [3], we now obtain concrete examples of the three-component spinor BEC ($F=1$, $m_F=1,0,-1$) for both antiferromagnetic (Na_{23}) [10] and ferromagnetic interaction cases. In the present spinor BEC the degenerate internal degrees of freedom play an

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essential role to determine the fundamental physical properties [4–6, 8, 11–13, 15]. There is a rich variety of topological defect structures, which are already predicted in the earlier studies [9] on the spinor BEC. An excellent algebraic representation of the $F=1$ BEC Hamiltonian to study the exact many-body states were constructed [2, 7], and they found that spin-exchange interactions cause a set of collective dynamic behavior of BEC. Recently, the spin wave excitation of spinor BECs, and the interaction between the spin waves have been studied [14, 16, 17]. In this letter, we shall study the quantum tunneling of spinor BECs in an optical lattice when the system being prepared in a squeezed coherent state.

2 Hamiltonian of Spinor BECs in an Optical Lattice

We consider the spinor BECs trapped in optical lattice, which are primarily governed by three types of two-body interactions including spin-change collision, magnetic dipole-dipole interaction and light-induced dipole-dipole interaction. The Hamiltonian of this system can be written by

$$H=\sum_n\int d\mathbf{r}\hat{\psi}_n^\dagger(\mathbf{r})\left[-\frac{\hbar^2\nabla^2}{2m}+V_L(\mathbf{r})\right]\hat{\psi}_n(\mathbf{r})+\sum_{n,m,n',m'}\int d\mathbf{r}d\mathbf{r}'\hat{\psi}_n^\dagger(\mathbf{r})\hat{\psi}_m^\dagger(\mathbf{r}')$$
$$\times[V_{nn'mm'}^{\text{coll}}(\mathbf{r},\mathbf{r}')+V_{nn'mm'}^{d-d}(\mathbf{r},\mathbf{r}')]\hat{\psi}_{m'}(\mathbf{r}')\hat{\psi}_{n'}(\mathbf{r})+H_B, \tag{1}$$

where $V_L(\mathbf{r})$ is the lattice potential, and the index $n,m,n',m'=-F,\dots,F$ denotes the Zeeman sublevels of the ground state of the atoms with angular momentum $\mathbf{F}$. $V_{nn'mm'}^{\text{coll}}(\mathbf{r},\mathbf{r}')$ stands for the two-body ground-state collisions, $V_{nn'mm'}^{d-d}(\mathbf{r},\mathbf{r}')$ include magnetic dipole-dipole interaction and light induced dipole-dipole interaction, and H_B describes the external magnetic interaction.

When the potential depth of the optical lattice is large enough, we can expand the spinor atomic field operator as $\hat{\psi}_m(\mathbf{r})=\sum_i\phi_i(\mathbf{r})\hat{C}_m(i)$, where $\phi_i(\mathbf{r})$ is the condensate wave function for the ith microtrap and the operators $\hat{C}_m(i)$ satisfy the general bosonic commutative relations $[\hat{C}_m(i),\hat{C}_n^\dagger(j)]=\delta_{mn}\delta_{ij}$. For the sake of simplicity, we consider the tight-binding approximation and consider only the spin-dependent terms. According to the Holstein–Primakoff transformation method, the spin operators $\hat{S}^\dagger$, $\hat{S}^-$ and $\hat{S}_z$ can be represented by Bose operators a and $a^\dagger$ with $[a,a^\dagger]=1$ and $[a,a]=[a^\dagger,a^\dagger]=0$, i.e., $\hat{S}^\dagger=(\sqrt{2S-a^\dagger a})a$, $\hat{S}^-=a^\dagger(\sqrt{2S-a^\dagger a})$, $\hat{S}_z=(S-a^\dagger a)$. If considering the hight-order terms (up to the third term), then (1) takes the form [14]

$$\begin{aligned}H&=\lambda_a' NS(S+1)-\gamma_B B_z SN+\gamma_B B_z\sum_i a_i^\dagger a_i\\&-\sum_i\sum_{j\neq i}(J_{ij}^{md}S^2-J_{ij}^{md}Sa_j^\dagger a_j-J_{ij}^{md}Sa_i^\dagger a_i+J_{ij}^{md}a_i^\dagger a_i a_j^\dagger a_j)\\&-\sum_i\sum_{j\neq i}(2J_{ij}Sa_i^\dagger a_j+2J_{ij}Sa_i a_j^\dagger)\\&+\frac{1}{2}\sum_i\sum_{j\neq i}J_{ij}(a_i^\dagger a_i^\dagger a_i a_j+a_i^\dagger a_j^\dagger a_j a_j+a_j a_i^\dagger a_i^\dagger a_i+a_j^\dagger a_j a_j a_i^\dagger)+\cdots,\end{aligned} \tag{2}$$

where $J_{ij} = J_{ij}^{ld} - \frac{1}{4}J_{ij}^{md}$, J_{ij}^{md} is a coefficient which describes magnetic dipole-dipole interaction and J_{ij}^{ld} is a coefficient which represents light induced dipole-dipole interaction. The collective spin operators are defined as $\hat{S}_i = \sum_{mn} \hat{C}_m^\dagger(i)\mathbf{F}_{mn}\hat{C}_n(i)$ with components $\hat{S}_i^{\{\pm,z\}}$, where $\mathbf{F}_{mn}$ is the matrix element of the angular moment $\mathbf{F}$. The direction of the magnetic field $\mathbf{B}$ is along the one-dimensional optical lattice, which we select as the quantization axis $\mathbf{z}$, and $\mathbf{B} = B_z\mathbf{z}$. The parameter $\gamma_B = -\mu_B g_F$ with μ_B being the Bohr magneton and g_F the Landé g factor.

3 Quantum Tunneling of Spinor BECs in an Optical Lattice

In this section, we shall study the tunneling of spinor Bose–Einstein condensates in optical lattice. The nonlinear interactions in Hamiltonian (2) generate the nonlinear magnetic excitations such as mixing of spin waves or magnetic solitons [14] depending on the initial setup of the state of the spin system. Here we are interested in the study of tunneling induced by the nonlinear interactions. The ideal case is that the spin system in the optical lattice should initially be prepared in a squeezed coherent state $|\{\psi_l\}\rangle = \Pi_l S(\beta_l)D(\alpha_l)|0\rangle$, where $S(\beta_l) = \exp[\frac{1}{2}\beta_l^* a^2 - \frac{1}{2}\beta_l a^{\dagger 2}]$, $\beta = |\beta|e^{i\theta}$ with $0 \le |\beta| < \infty, 0 \le \theta \le 2\pi$ and $D(\alpha_l) = \exp[\alpha_l a^\dagger - \alpha_l^* a]$. The vacuum state $|0\rangle$ is the ground state of the BEC in the optical lattice, i.e., $|0\rangle = |GS\rangle = |N, N\rangle$.

Under the spin coherent state and using the time-dependent variation principle, the nonlinear motion equation of atomic number $\psi_l = \langle\psi|a_l^\dagger a_l|\psi\rangle$ on the lattice l can be derived as

$$
\begin{aligned}
i\hbar\frac{\partial\psi_l}{\partial t} &= (\gamma B + 4SJ^{MD})(\psi_l \cosh|\beta| + e^{i\theta}\sinh|\beta|\psi_l^*) \\
&\quad - 2J^{MD}(\psi_l \cosh|\beta| + e^{i\theta}\sinh|\beta|\psi_l^*)[(\cosh^2|\beta| + \sinh^2|\beta|)(|\psi_{l+1}|^2 + |\psi_{l-1}|^2) \\
&\quad + 2\sinh^2|\beta| + \cosh|\beta|\sinh|\beta|(e^{i\theta}\psi_{l+1}^{*2} + e^{i\theta}\psi_{l-1}^{*2} + e^{-i\theta}\psi_{l+1}^2 + e^{-i\theta}\psi_{l-1}^2)] \\
&\quad - 4SJ[\cosh|\beta|(\psi_{l+1} + \psi_{l-1}) + e^{i\theta}\sinh|\beta|(\psi_{l+1}^* + \psi_{l-1}^*)] \\
&\quad + 2J[(\cosh^2|\beta| + \sinh^2|\beta|)|\psi_l|^2 + \sinh^2|\beta| + e^{i\theta}\cosh|\beta|\sinh|\beta|\psi_l^{*2} \\
&\quad + e^{-i\theta}\cosh|\beta|\sinh|\beta|\psi_l^2][\cosh|\beta|(\psi_{l+1} + \psi_{l-1}) + e^{i\theta}\sinh|\beta|(\psi_{l+1}^* + \psi_{l-1}^*)] \\
&\quad + J[\cosh^2|\beta|\psi_l^2 + e^{i\theta}\cosh|\beta|\sinh|\beta|(2|\psi_l|^2 + 1) + e^{2i\theta}\sinh^2|\beta|\psi_l^{*2}] \\
&\quad \times [\cosh|\beta|(\psi_{l+1}^* + \psi_{l-1}^*) + e^{-i\theta}\sinh|\beta|(\psi_{l+1} + \psi_{l-1})] \\
&\quad + Je^{2i\theta}\cosh|\beta|\sinh^3|\beta|(\psi_{l+1}^{*3} + \psi_{l-1}^{*3}) + Je^{-i\theta}\cosh^2|\beta|\sinh|\beta|(\psi_{l+1}^3 + \psi_{l-1}^3) \\
&\quad + J(\cosh^3|\beta| + 2\cosh|\beta|\sinh^3|\beta|)(|\psi_{l+1}|^2\psi_{l+1} + |\psi_{l-1}|^2\psi_{l-1}) \\
&\quad + J(2e^{i\theta}\cosh^2|\beta|\sinh|\beta| + e^{i\theta}\sinh^3|\beta|)(|\psi_{l+1}|^2\psi_{l+1}^* + |\psi_{l-1}|^2\psi_{l-1}^*) \\
&\quad + J(e^{i\theta}\cosh^2|\beta|\sinh|\beta| + 2e^{i\theta}\sinh^3|\beta|)(\psi_{l+1}^* + \psi_{l-1}^*) \\
&\quad + J(\cosh|\beta|\sinh^2|\beta| + 2\cosh|\beta|\sinh^3|\beta|)(\psi_{l+1} + \psi_{l-1}),
\end{aligned} \tag{3}
$$

where $J = J^{LD} - \frac{J^{MD}}{4}$ with J^{MD} and J^{LD} are the average values of J_{ij}^{md} and J_{ij}^{ld}.

Similarly, we can obtain the motion equation of atomic number $\mu_{l+1} = \langle\psi|a^{\dagger}_{l+1}a_{l+1}|\psi\rangle$ and $\nu_{l-1} = \langle\psi|a^{\dagger}_{l-1}a_{l-1}|\psi\rangle$ on the lattice $l+1$ and $l-1$, respectively:

$$
\begin{aligned}
i\hbar\frac{\partial\mu_{l+1}}{\partial t} &= (\gamma B + 4SJ^{MD})(\mu_l\cosh|\beta| + e^{i\theta}\sinh|\beta|\mu^*_{l+1})\\
&\quad - 2J^{MD}(\mu_{l+1}\cosh|\beta| + e^{i\theta}\sinh|\beta|\mu^*_{l+1})\\
&\quad \times[(\cosh^2|\beta| + \sinh^2|\beta|)(|\mu_{l+2}|^2 + |\mu_l|^2)\\
&\quad + 2\sinh^2|\beta| + \cosh|\beta|\sinh|\beta|(e^{i\theta}\mu^{*2}_{l+2} + e^{i\theta}\mu^{*2}_l + e^{-i\theta}\mu^2_{l+2} + e^{-i\theta}\mu^2_l)]\\
&\quad - 4SJ[\cosh|\beta|(\mu_{l+2} + \mu_l) + e^{i\theta}\sinh|\beta|(\mu^*_{l+2} + \mu^*_l)]\\
&\quad + 2J[(\cosh^2|\beta| + \sinh^2|\beta|)|\mu_{l+1}|^2 + \sinh^2|\beta| + e^{i\theta}\cosh|\beta|\sinh|\beta|\mu^{*2}_{l+1}\\
&\quad + e^{-i\theta}\cosh|\beta|\sinh|\beta|\mu^2_{l+1}][\cosh|\beta|(\mu_{l+2} + \mu_l) + e^{i\theta}\sinh|\beta|(\mu^*_{l+2} + \mu^*_l)]\\
&\quad + J[\cosh^2|\beta|\mu^2_{l+1} + e^{i\theta}\cosh|\beta|\sinh|\beta|(2|\mu_{l+1}|^2 + 1) + e^{2i\theta}\sinh^2|\beta|\mu^{*2}_{l+1}]\\
&\quad \times[\cosh|\beta|(\mu^*_{l+2} + \mu^*_l) + e^{-i\theta}\sinh|\beta|(\mu_{l+2} + \mu_l)]\\
&\quad + Je^{2i\theta}\cosh|\beta|\sinh^3|\beta|(\mu^{*3}_{l+2} + \mu^{*3}_l) + Je^{-i\theta}\cosh^2|\beta|\sinh|\beta|(\mu^3_{l+2} + \mu^3_l)\\
&\quad + J(\cosh^3|\beta| + 2\cosh|\beta|\sinh^3|\beta|)(|\mu_{l+2}|^2\mu_{l+2} + |\mu_l|^2\mu_l)\\
&\quad + J(2e^{i\theta}\cosh^2|\beta|\sinh|\beta| + e^{i\theta}\sinh^3|\beta|)(|\mu_{l+2}|^2\mu^*_{l+2} + |\mu_l|^2\mu^*_l)\\
&\quad + J(e^{i\theta}\cosh^2|\beta|\sinh|\beta| + 2e^{i\theta}\sinh^3|\beta|)(\mu^*_{l+2} + \mu^*_l)\\
&\quad + J(\cosh|\beta|\sinh^2|\beta| + 2\cosh|\beta|\sinh^3|\beta|)(\mu_{l+2} + \mu_l),
\end{aligned}
\tag{4}
$$

and

$$
\begin{aligned}
i\hbar\frac{\partial\nu_{l-1}}{\partial t} &= (\gamma B + 4SJ^{MD})(\nu_{l-1}\cosh|\beta| + e^{i\theta}\sinh|\beta|\nu^*_{l-1})\\
&\quad - 2J^{MD}(\nu_{l-1}\cosh|\beta| + e^{i\theta}\sinh|\beta|\nu^*_{l-1})[(\cosh^2|\beta| + \sinh^2|\beta|)(|\nu_l|^2 + |\nu_{l-2}|^2)\\
&\quad + 2\sinh^2|\beta| + \cosh|\beta|\sinh|\beta|(e^{i\theta}\nu^{*2}_l + e^{i\theta}\nu^{*2}_{l-2} + e^{-i\theta}\nu^2_l + e^{-i\theta}\nu^2_{l-2})]\\
&\quad - 4SJ[\cosh|\beta|(\nu_l + \nu_{l-2}) + e^{i\theta}\sinh|\beta|(\nu^*_l + \nu^*_{l-2})]\\
&\quad + 2J[(\cosh^2|\beta| + \sinh^2|\beta|)|\nu_{l-1}|^2 + \sinh^2|\beta| + e^{i\theta}\cosh|\beta|\sinh|\beta|\nu^{*2}_{l-1}\\
&\quad + e^{-i\theta}\cosh|\beta|\sinh|\beta|\nu^2_{l-1}][\cosh|\beta|(\nu_l + \nu_{l-2}) + e^{i\theta}\sinh|\beta|(\nu^*_l + \nu^*_{l-2})]\\
&\quad + J[\cosh^2|\beta|\nu^2_{l-1} + e^{i\theta}\cosh|\beta|\sinh|\beta|(2|\nu_{l-1}|^2 + 1) + e^{2i\theta}\sinh^2|\beta|\nu^{*2}_{l-1}]\\
&\quad \times[\cosh|\beta|(\nu^*_l + \nu^*_{l-2}) + e^{-i\theta}\sinh|\beta|(\nu_l + \nu_{l-2})]\\
&\quad + Je^{2i\theta}\cosh|\beta|\sinh^3|\beta|(\nu^{*3}_l + \nu^{*3}_{l-2}) + Je^{-i\theta}\cosh^2|\beta|\sinh|\beta|(\nu^3_l + \nu^3_{l-2})\\
&\quad + J(\cosh^3|\beta| + 2\cosh|\beta|\sinh^3|\beta|)(|\nu_l|^2\nu_l + |\nu_{l-2}|^2\nu_{l-2})\\
&\quad + J(2e^{i\theta}\cosh^2|\beta|\sinh|\beta| + e^{i\theta}\sinh^3|\beta|)(|\nu_l|^2\nu^*_l + |\nu_{l-2}|^2\nu^*_{l-2})\\
&\quad + J(e^{i\theta}\cosh^2|\beta|\sinh|\beta| + 2e^{i\theta}\sinh^3|\beta|)(\nu^*_l + \nu^*_{l-2})\\
&\quad + J(\cosh|\beta|\sinh^2|\beta| + 2\cosh|\beta|\sinh^3|\beta|)(\nu_l + \nu_{l-2}).
\end{aligned}
\tag{5}
$$

According to (3–5), we see that $\frac{\partial}{\partial t}(\psi_l - \mu_{l+1}) \neq \frac{\partial}{\partial t}(\psi_l - \nu_{l-1})$, which means that the tunnelings between lattices l and $l+1$, l and $l-1$ are general different. In particular, when the optical lattice is infinitely long and the spin excitations are in the long-wavelength limit, one has $\psi_l = \psi_{l+1} = \psi_{l-1}$, $\mu_l = \mu_{l+1} = \mu_{l-1}$, and $\nu_l = \nu_{l+1} = \nu_{l-1}$ in the continuum limit approximation, this shows that there does not exist the tunneling effect between lattices l and $l+1$, and that l and $l-1$. Correspondingly, the magnetic soliton appears [14].

4 Conclusions

In summary, we have studied quantum tunneling of spinor BECs in an optical lattice, it is found that, when the system being prepared in an squeezed coherent state, there exist the quantum tunneling between lattices l and $l+1$, l and $l-1$, respectively. In particular, when the optical lattice is infinitely long and the spin excitations are in the long-wavelength limit, the quantum tunneling disappears between lattices l and $l+1$, and that l and $l-1$, in this case the magnetic soliton appears.

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